Real Analysis tutorial class 1

Example 0.1. *Given $f : \mathbb{R} \to \mathbb{R}$. The set where f is continuous is a G_δ set.*

Proof. Denote A to be the set where f is continuous for simplicity. By definition,

$$\begin{aligned} A &= \bigcap_{n\in\mathbb{N}} \{a : \exists \delta = \delta(a) > 0,\ |f(x) - f(y)| < n^{-1},\ \forall x, y \in V_\delta(a)\} \\ &= \bigcap_{n=1}^{\infty} A_n \end{aligned}$$

We show that each A_n is open. Let $a \in A_n$, we obtain $\delta > 0$ possibly depending on a. Let $\epsilon < \delta/2$. Then for any $b \in V_\epsilon(a)$, we have

$$V_{\delta/2}(b) \subset V_\delta(a).$$

Hence, for all $x, y \in V_{\delta/2}(b)$, $|f(x) - f(y)| < n^{-1}$. This show the openness.

Here we denote $V_\epsilon(x) = \{y : |x - y| < \epsilon\}$. □

Example 0.2. *If f is continuous on $\mathbb{R}$, then $f^{-1}(B)$ is Borel set if B is Borel.*

Proof. Denote

$$M = \{A \in P(\mathbb{R}) : f^{-1}(A)\}.$$

It suffices to show that M is a σ-algebra containning all open sets in $\mathbb{R}$.

Clearly, $\mathbb{R} \in M$. If $A \in M$, then $f^{-1}(A^c) = \left(f^{-1}(A)\right)^c$ is Borel. Hence $A^c \in M$.

If $A_i, i \in \mathbb{N}$ is inside M, then $f^{-1}(\cup A_i) = \cup f^{-1}(A_i)$ is Borel. $\cup_{i\in\mathbb{N}} A_i \in M$. Therefore M is a σ-algebra.

By continuity, M contains all open sets. This complete the proof. □